
UNIVERSITY *of* GUELPH

Final Project Report

ENGG*6090: Numerical Methods for Engineers

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Abstract

This report presents a numerical method to estimate heat transfer in the system. For the estimation of heat flow, temperature distribution throughout the system requires. Therefore, the main aim of the project is to find out the temperature distribution and for which a deterministic relationship of temperature is developed by Taylor series expansion and Finite Difference Method (FDM). An application is made with the use of MATLAB App Designer and a 2D numerical example is solved using FDM. A good accordance can be observed in a comparison of the results of the MATLAB App and the results from the FlexPDE – a finite element model builder for solving partial differential equation problems.

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Table of Symbols

Sr. No.	Symbols	Unit	Interpretation
1	q	$\frac{W}{m^2}$	Heat flux
2	k	$\frac{W}{mK}$	Thermal conductivity
3	A	m^2	Area
4	T	$^{\circ}C$	Temperature
5	h	$\frac{W}{m^2K}$	Heat transfer coefficient
6	T_s	$^{\circ}C$	Surface temperature
7	T_{∞}	$^{\circ}C$	Atmospheric temperature
8	g	$\frac{W}{m^3}$	Volumetric heat generation
9	ρ	$\frac{kg}{m^3}$	Density
10	C	$\frac{J}{kg * K}$	Specific heat capacity
11	t	s	Time
12	\dot{e}	J	Energy

Introduction:

Heat transfer predicts the transfer of heat energy from one body to another by the means of temperature difference. Heat transfer takes place by three different type of modes, and they are: Conduction, Convection and Radiation. However, in this study radiation mode of heat transfer is neglected and assumed that the heat transfer occurs only because of conduction and convection. Conduction heat transfer is governed by the Fourier's law equation which can be written as, $q = -k * A * \nabla T$, where, q is the heat flux, k is the thermal conductivity of the material, A is the cross-sectional area and ∇T is the temperature gradient. Convection heat transfer is governed by the Newton's law of cooling which is given as: $q = h * A * (T_s - T_\infty)$, where, q is the heat flux, h is the heat transfer coefficient, A is the cross-sectional area, T_s is the temperature of the surface of the body and T_∞ is the atmospheric temperature [1].

For this project the computational domain is the furnace as shown in figure 1(a) [2]. However, for the simplicity the computational domain is selected as shown in figure 1(b).

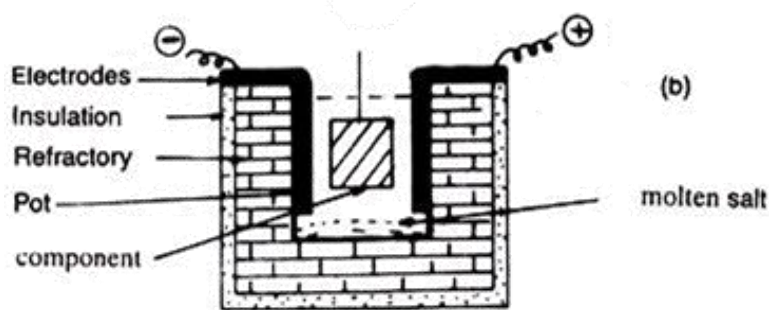


Figure 1(a): Cross Section of the Furnace [2]

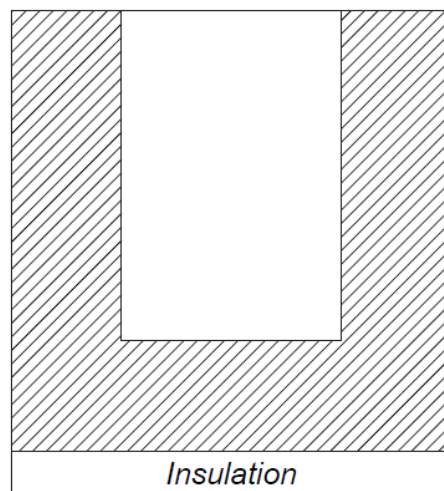


Figure 1(b): Cross Section of the Furnace

Domain:

Consider the cross section of the furnace which is insulated from the bottom and in contact with the air from the other side. The dimensions are in meters as shown in the figure 2. It is surrounded by the atmospheric air with the temperature (T_∞) and the heat transfer coefficient of the air is (h).

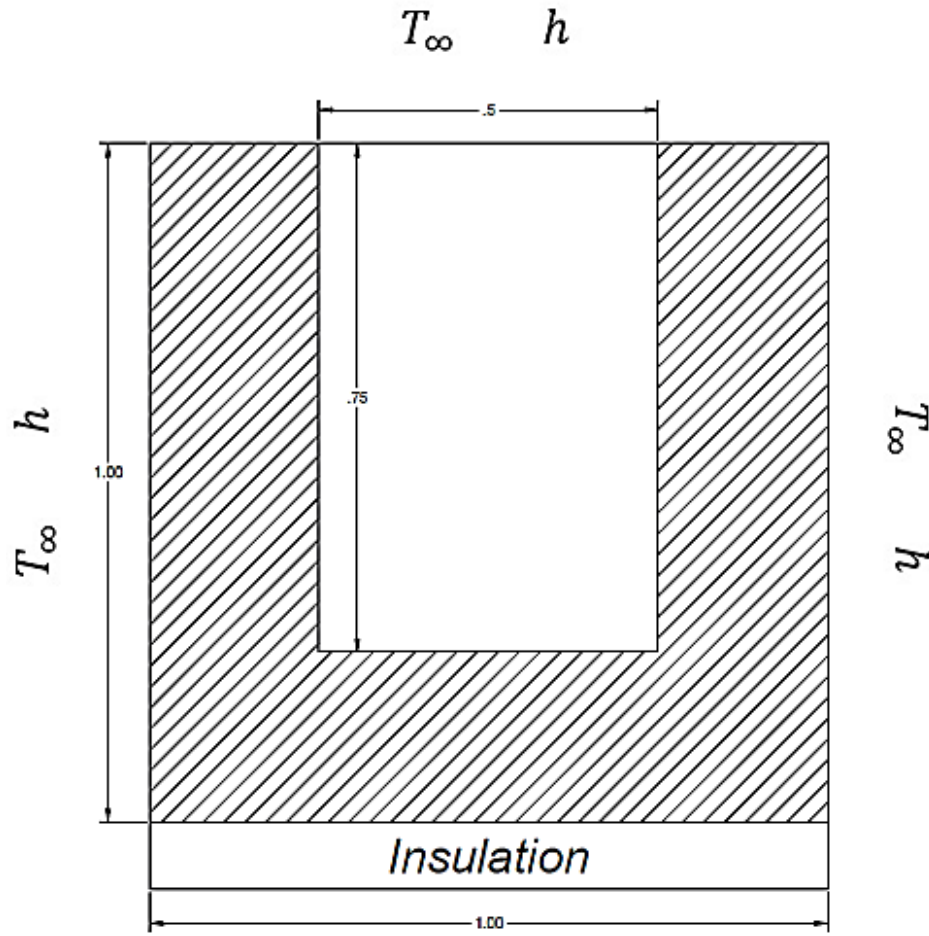


Figure 2: Domain

Assumptions:

- 2D steady state heat transfer – temperature does not vary significantly along z direction when compared with x and y direction
- Thermal conductivity is same throughout the concrete structure, i.e. the system is homogeneous and isotropic.
- Core temperature of the furnace is 1300°C and is same at every point in the core.
- No thermal radiation heat transfer.
- No heat generation, i.e. $g = 0$

Boundary Conditions:

$$\begin{aligned} @y=0, x; & \quad \frac{\partial T}{\partial x} = 0 \\ @x=0, y; & \quad -k * \frac{\partial T}{\partial x} = h * (T_s - T_\infty) \\ @y=L, x; & \quad -k * \frac{\partial T}{\partial x} = h * (T_s - T_\infty) \\ @x=L, y; & \quad -k * \frac{\partial T}{\partial x} = h * (T_s - T_\infty) \end{aligned}$$

Governing equation:

The governing equation for 3D heat transfer conduction is given by:

$$k_x \frac{\partial^2 T}{\partial x^2} + k_y \frac{\partial^2 T}{\partial y^2} + k_z \frac{\partial^2 T}{\partial z^2} + g = \rho * C * \frac{\partial T}{\partial t}$$

Where,

- k_x = thermal conductivity of the concrete material in x direction, W/mK
- k_y = thermal conductivity of the concrete material in y direction, W/mK
- k_z = thermal conductivity of the concrete material in z direction, W/mK
- g = volumetric rate of heat generation, W/m³
- ρ = density of the concrete material, kg/m³
- C = specific heat capacity of the material, J/kgK
- t = time, s

According to the assumptions the governing equation for the selected domain is reduced to:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Finite Difference Analysis:

For estimating the heat flow in the domain, temperature distribution is to be known. FDM (Finite Difference Method) a numerical technique is used to find out the temperature distribution which provides an approximate result. In FDM, the domain is subdivided into number of small regions and temperature is determined at the discrete points of the regions which are called as nodes. Each node depicts the temperature of that corresponding point. Using the boundary conditions, algebraic equations are formed to define the relationship between the adjacent nodes. The resolution of the result depends on the number of nodes. Hence, by increasing the number of nodes high resolution result can be achieved [3]. For this project, the whole system is divided into a nodal network of 100x100 nodes as shown in figure 3. However, in the application developed by MATLAB App Designer the user can select the number of nodes which is multiple of 4.

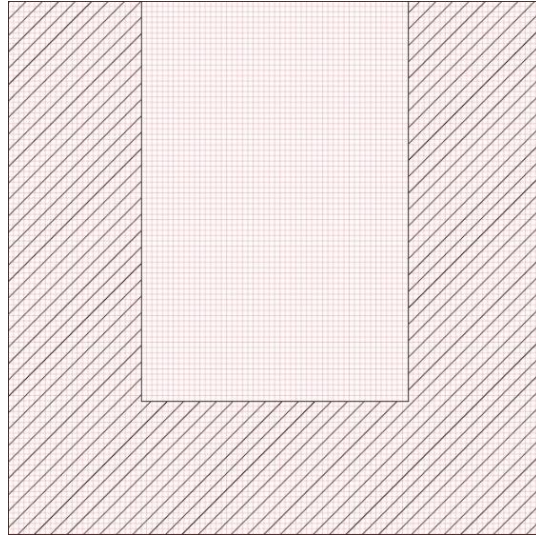


Figure 3: Nodal Point Network

The nodes are assumed to be on the intersection of the grid lines. The computational node at which the temperature is calculated that particular node is designated by the indices (m,n) as shown in figure 4. The distance between adjacent nodes is Δx and Δy in x and y direction respectively. However, for the case of simplicity $\Delta x = \Delta y$.

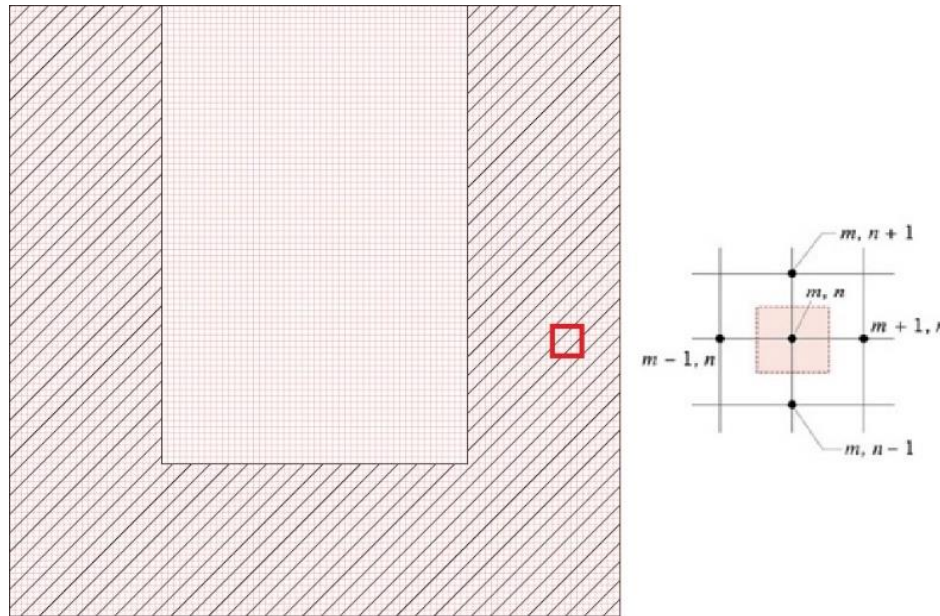


Figure 4: Interior node [4]

In FDM, the derivatives are substituted by differences which is given as below:

$$\frac{df(x)}{dx} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{df(x)}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta f(x)}{\Delta x}$$

Therefore,

$$\frac{\partial T}{\partial x} \Big|_{m-\frac{1}{2}} = \frac{T_m - T_{m-1}}{\Delta x}$$

$$\frac{\partial T}{\partial x} \Big|_{m+\frac{1}{2}} = \frac{T_{m+1} - T_m}{\Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} \Big|_m = \frac{\frac{\partial T}{\partial x} \Big|_{m+\frac{1}{2}} - \frac{\partial T}{\partial x} \Big|_{m-\frac{1}{2}}}{\Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} \Big|_m = \frac{\frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

$$\frac{\partial^2 T}{\partial x^2} \Big|_m = \frac{T_{m-1} - 2 * T_m + T_{m+1}}{\Delta x^2}$$

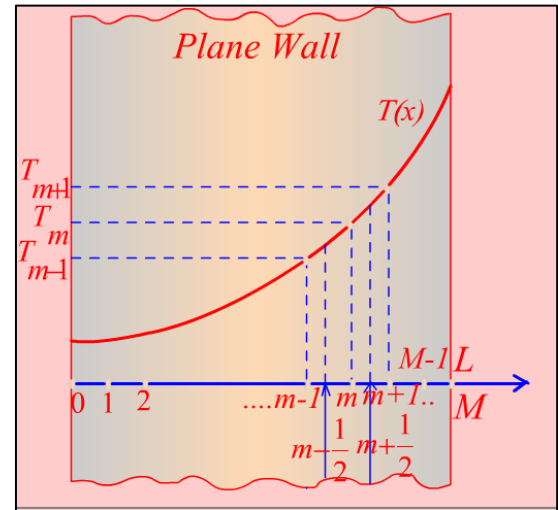
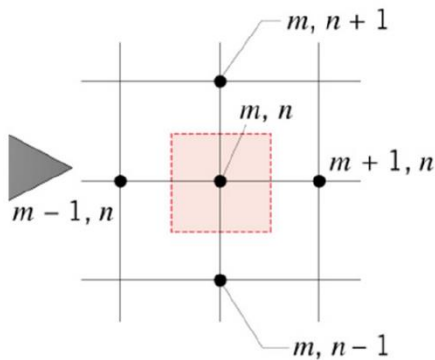


Figure 5: Plane Wall [5]

Finite difference equation for every node for unknown temperature is obtain by using the energy balance method. For any interior node as shown in given figure 6 the governing equation $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$, can be written in terms of Finite Difference Equation.



$$\left(\frac{T_{m-1,n} - 2 * T_{m,n} + T_{m+1,n}}{\Delta x^2} \right) + \left(\frac{T_{m,n-1} - 2 * T_{m,n} + T_{m,n+1}}{\Delta y^2} \right) = 0$$

As $\Delta x = \Delta y$,

$$T_{m,n} = \frac{T_{m-1,n} + T_{m+1,n} + T_{m,n-1} + T_{m,n+1}}{4}$$

Figure 6: Interior node [4]

Top corners are surrounded by the atmospheric air from two sides and the finite difference equation for top corners can be obtain by energy balance equation and its given as:

$$\dot{e}_{in} - \dot{e}_{out} = 0$$

Therefore, for top corners as shown in figure 7,

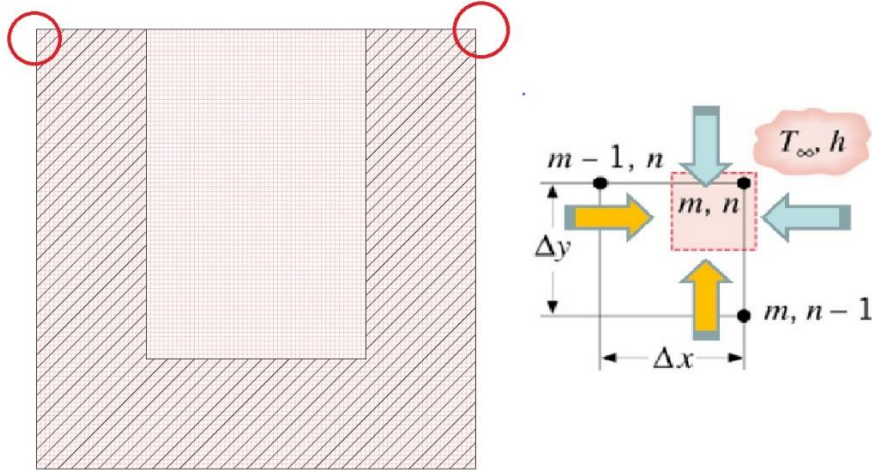


Figure 7: Top corner nodes [4]

For top right corner,

$$\sum q = 0$$

$$k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h \left(\frac{\Delta x}{2} * l \right) (T_{\infty} - T_{m,n}) + h \left(\frac{\Delta y}{2} * l \right) (T_{\infty} - T_{m,n}) = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m-1,n} + T_{m,n-1} + \frac{2h\Delta x T_{\infty}}{k}}{2 \left(\frac{h\Delta x}{k} + 1 \right)}$$

For top left corner,

$$\sum q = 0$$

$$k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h \left(\frac{\Delta x}{2} * l \right) (T_{\infty} - T_{m,n}) + h \left(\frac{\Delta y}{2} * l \right) (T_{\infty} - T_{m,n}) = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m+1,n} + T_{m,n-1} + \frac{2h\Delta x T_{\infty}}{k}}{2 \left(\frac{h\Delta x}{k} + 1 \right)}$$

Similarly, finite difference equation for the bottom corners can be obtain. However, in this case the corners are surrounded by only one side with the atmospheric air and is insulated from the other side. Therefore, for bottom corners as shown in figure 8 the finite difference equation can be obtained as:

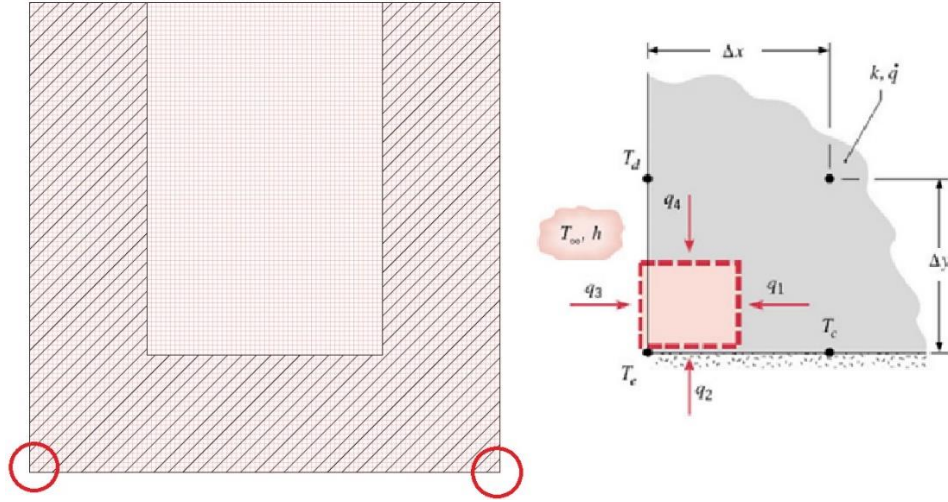


Figure 8: Bottom corner nodes [4]

For bottom right corner,

$$\sum q = 0$$

$$k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + h \left(\frac{\Delta y}{2} * l \right) (T_{\infty} - T_{m,n}) = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m-1,n} + T_{m,n+1} + \frac{h\Delta x T_{\infty}}{k}}{\frac{h\Delta x}{k} + 1}$$

For bottom left corner,

$$\sum q = 0$$

$$k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + h \left(\frac{\Delta y}{2} * l \right) (T_{\infty} - T_{m,n}) = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m+1,n} + T_{m,n+1} + \frac{h\Delta x T_{\infty}}{k}}{\frac{h\Delta x}{k} + 1}$$

Now, for the sides of the domain which are in contact with the atmospheric air, the finite difference equation can be obtain as below:

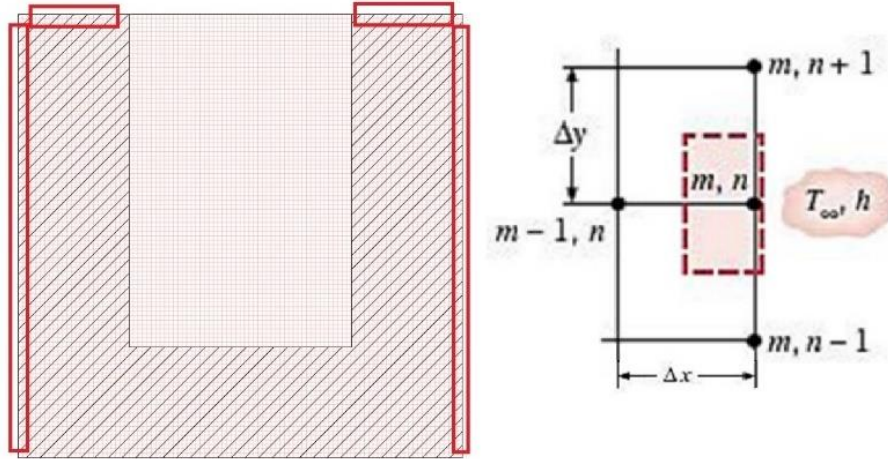


Figure 9: Left, Top and Right sides [4]

For right side,

$$\sum q = 0$$

$$k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k \left(\frac{2\Delta y}{2} * l \right) \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + h \left(\frac{2\Delta y}{2} * l \right) (T_{\infty} - T_{m,n}) = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m,n-1} + T_{m,n+1} + 2T_{m-1,n} + \frac{2h\Delta x T_{\infty}}{k}}{4 + \frac{2h\Delta x}{k}}$$

Similarly, for left side

$$\sum q = 0$$

$$k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + k \left(\frac{\Delta x}{2} * l \right) \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k \left(\frac{2\Delta y}{2} * l \right) \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + h \left(\frac{2\Delta y}{2} * l \right) (T_{\infty} - T_{m,n}) = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m,n-1} + T_{m,n+1} + 2T_{m+1,n} + \frac{2h\Delta x T_{\infty}}{k}}{4 + \frac{2h\Delta x}{k}}$$

Now for top sides

$$\sum q = 0$$

$$k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k \left(\frac{2\Delta x}{2} * l \right) \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + h \left(\frac{2\Delta x}{2} * l \right) (T_{\infty} - T_{m,n}) = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m-1,n} + T_{m+1,n} + 2T_{m,n-1} + \frac{2h\Delta x T_{\infty}}{k}}{4 + \frac{2h\Delta x}{k}}$$

For the bottom side of the domain which is insulated and the finite difference equation is derived as below:

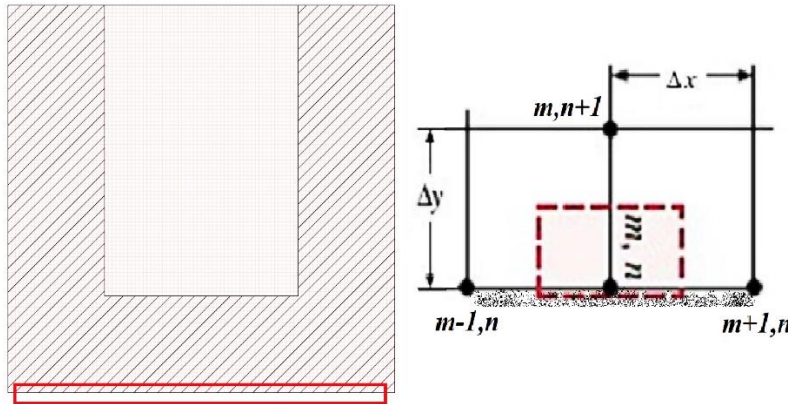


Figure 10: Bottom side [4]

$$\sum q = 0$$

$$k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k \left(\frac{\Delta y}{2} * l \right) \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k \left(\frac{2\Delta x}{2} * l \right) \frac{T_{m,n+1} - T_{m,n}}{\Delta y} = 0$$

as $\Delta x = \Delta y$;

$$T_{m,n} = \frac{T_{m-1,n} + T_{m+1,n} + 2T_{m,n+1}}{4}$$

Now for all the nodes that lies on the core boundary shown in figure 11, the temperature is substituted as $T_{m,n} = 1300$ ($^{\circ}\text{C}$).

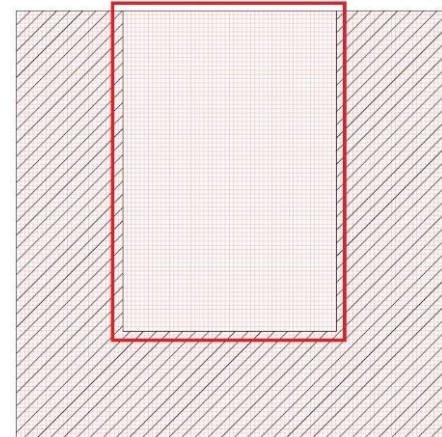


Figure 11: Nodes of the Core [4]

MATLAB App:

Using all the derived finite difference equation Application is developed using MATLAB App Designer and its user interface is shown in figure 12.

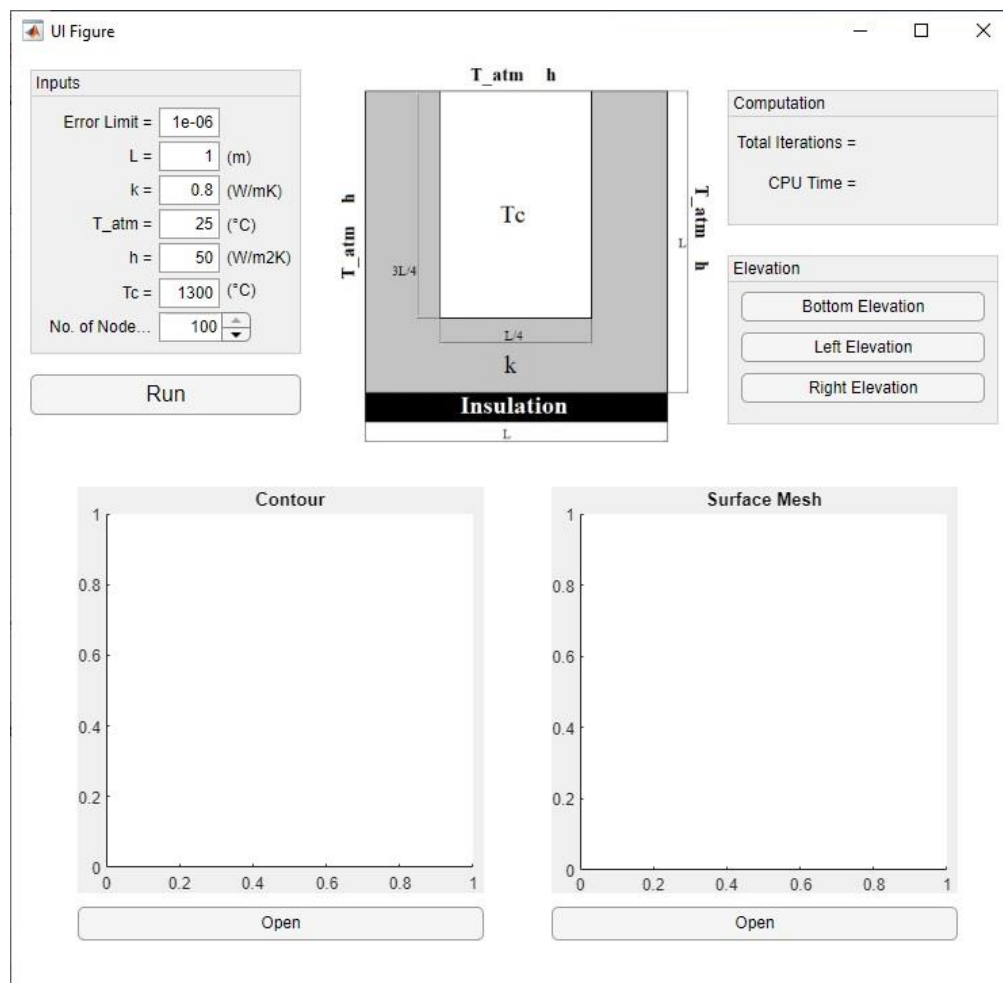


Figure 12: User Interface of Application

On clicking the “Run” button the script runs and gives the output result of temperature distribution throughout the selected domain. For validating the result from the app, FlexPDE – a software which works on Finite Element Method for obtaining numerical solution of partial differential equations in 2D or 3D. The Comparison between the results from both the software will be discussed further.

Result:

For the results the application provides an output as temperature plots and for their validation the results of the MATLAB app and FlexPDE were compared. Providing the same input in both the softwares and the results obtained are represented below:

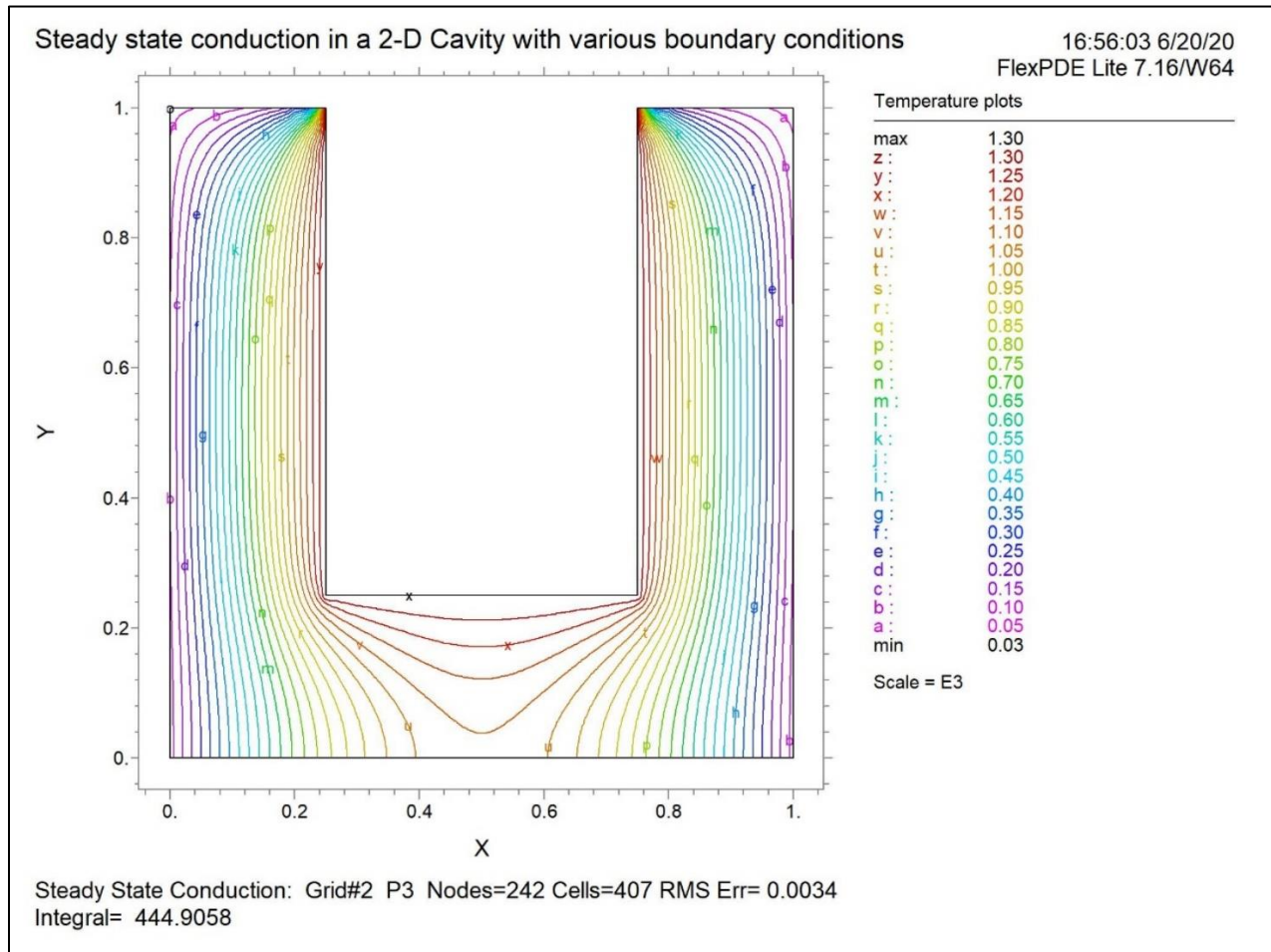


Figure 13: Contour from FlexPDE

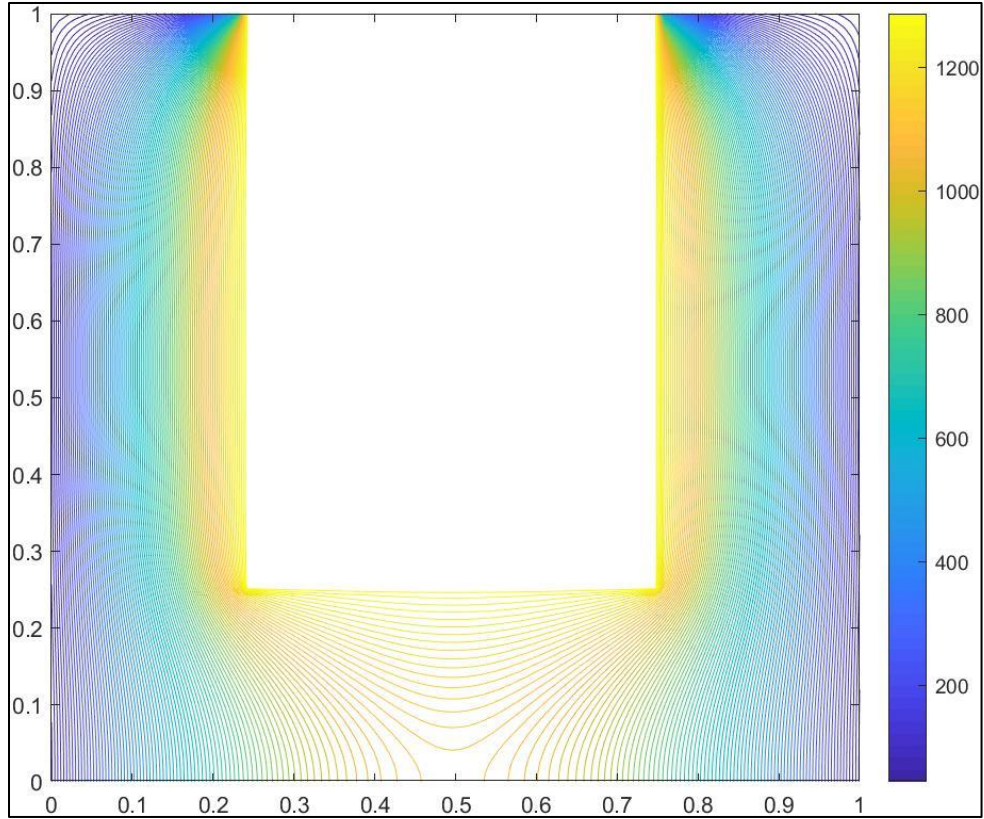


Figure 14: Contour from MATLAB App

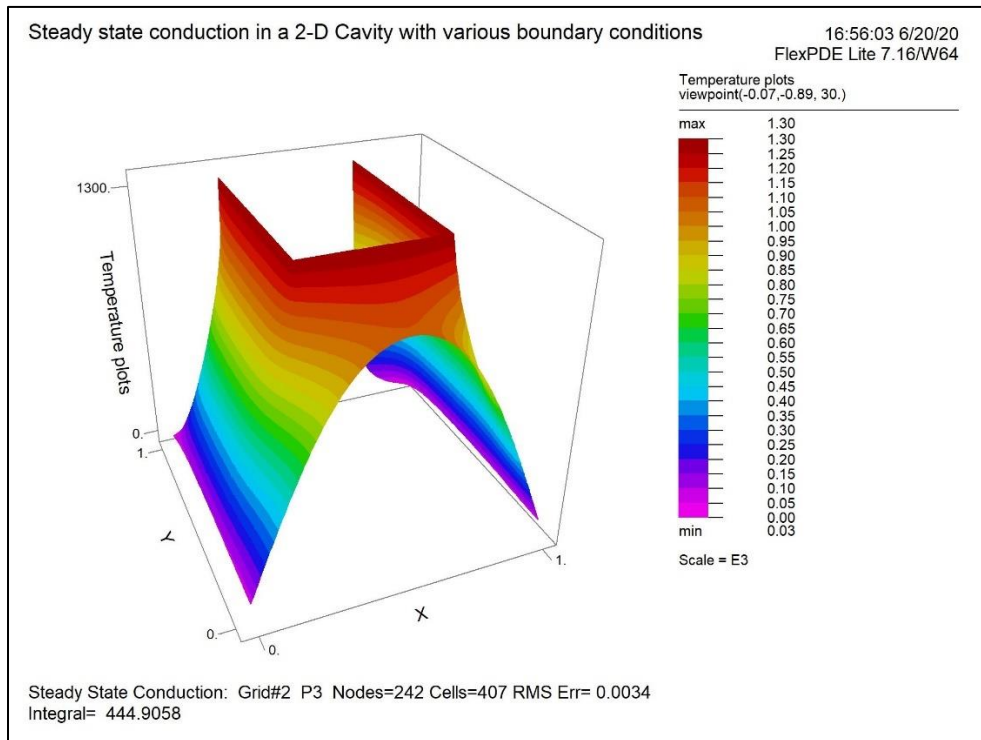


Figure 15: Surface Mesh from FlexPDE

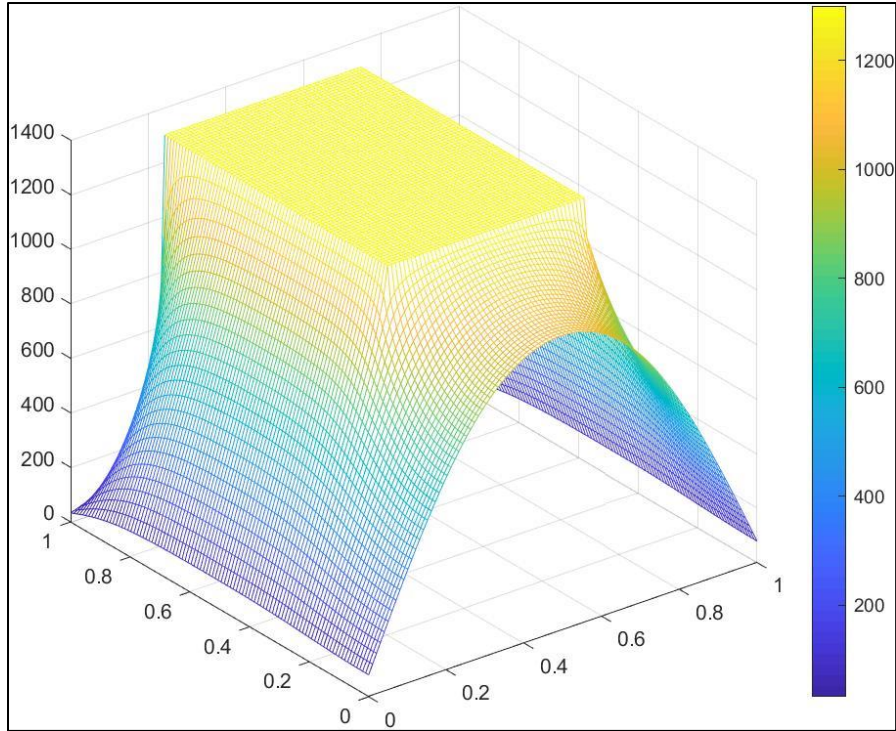


Figure 16: Surface Mesh from MATLAB App

From the above results it can be noted that the temperature ranges from around 30°C to 1300°C and the results from the MATLAB App matches the results from FlexPDE as well. Now, for further verification the elevation of temperature plot of the left, right and bottom side of the domain is compared as shown in figures 17-22 respectively.

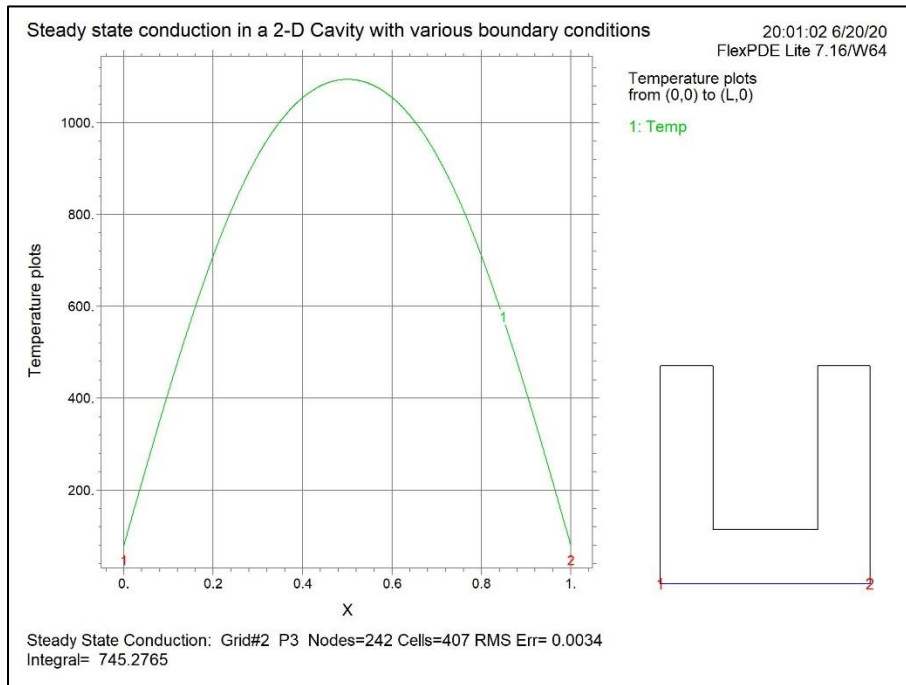


Figure 17: Elevation of the bottom edge from FlexPDE

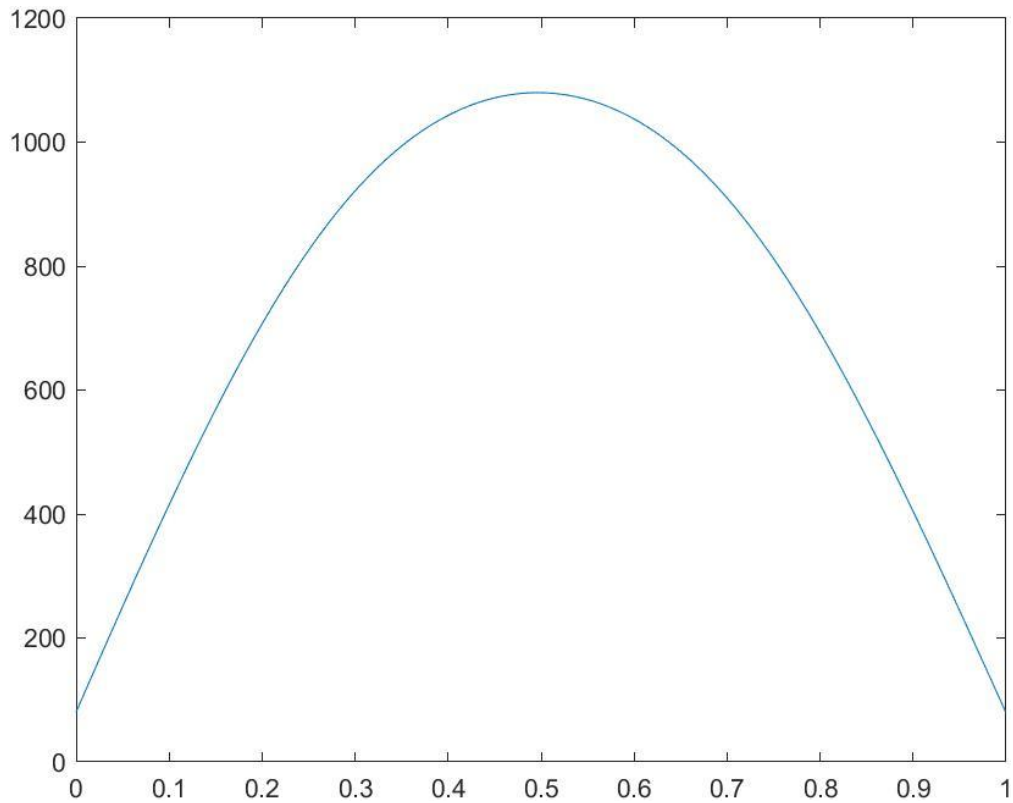


Figure 18: Elevation of the bottom edge from MATLAB App

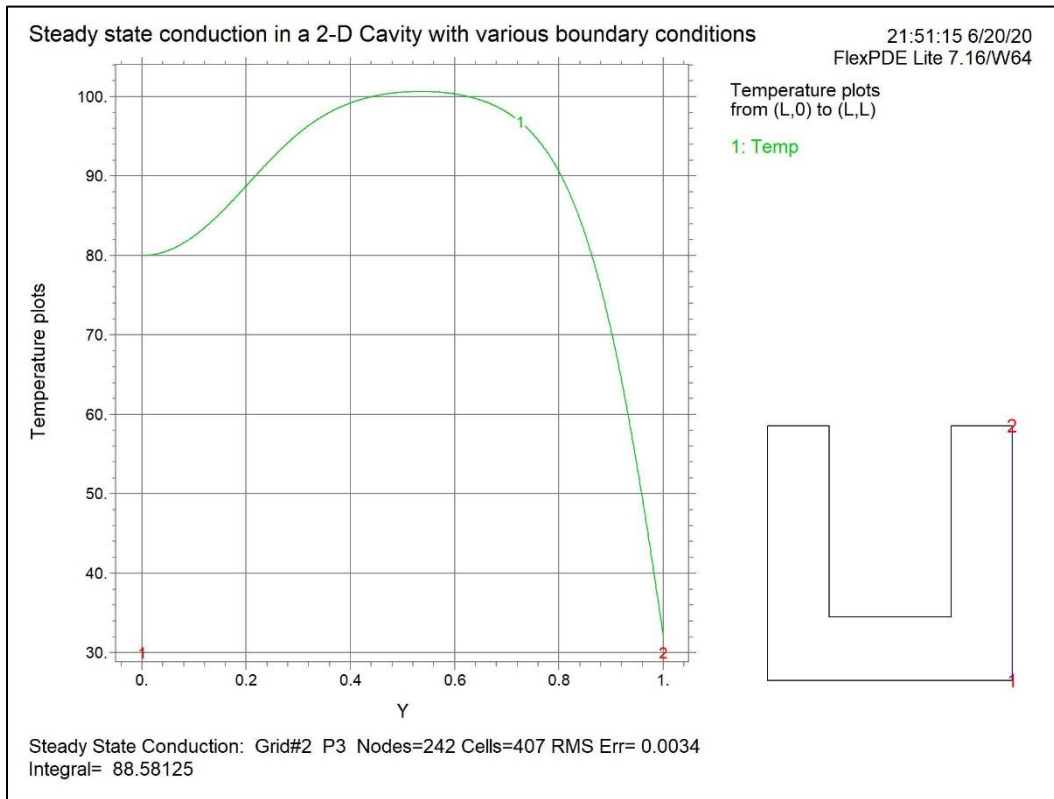


Figure 19: Elevation of the right edge from FlexPDE

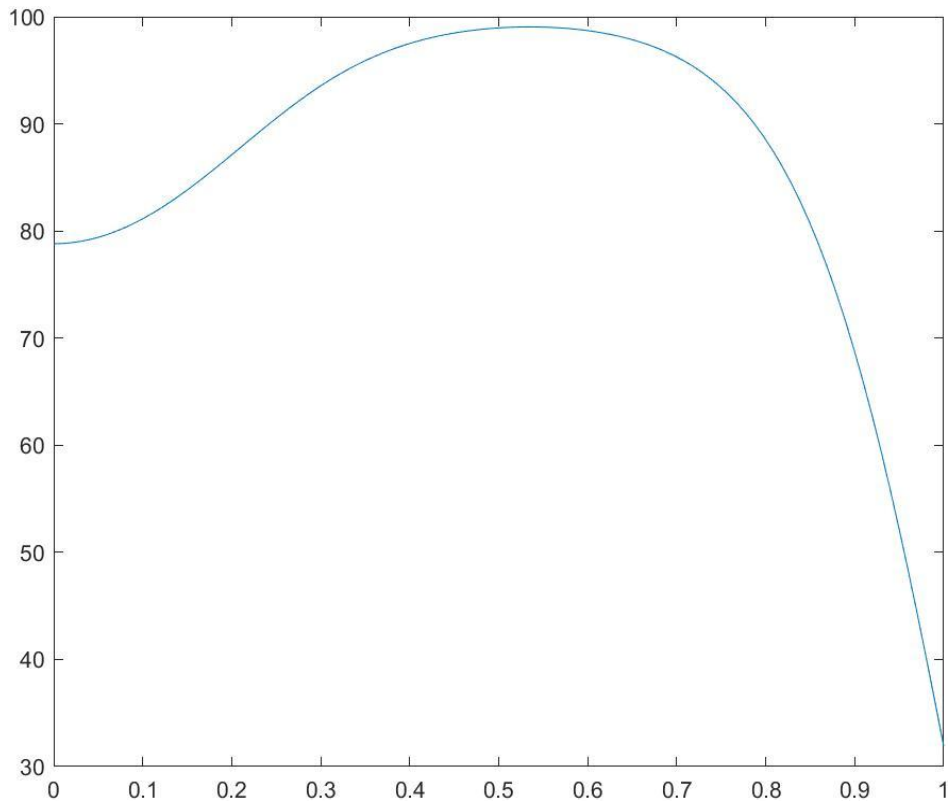


Figure 20: Elevation of the right edge from MATLAB App

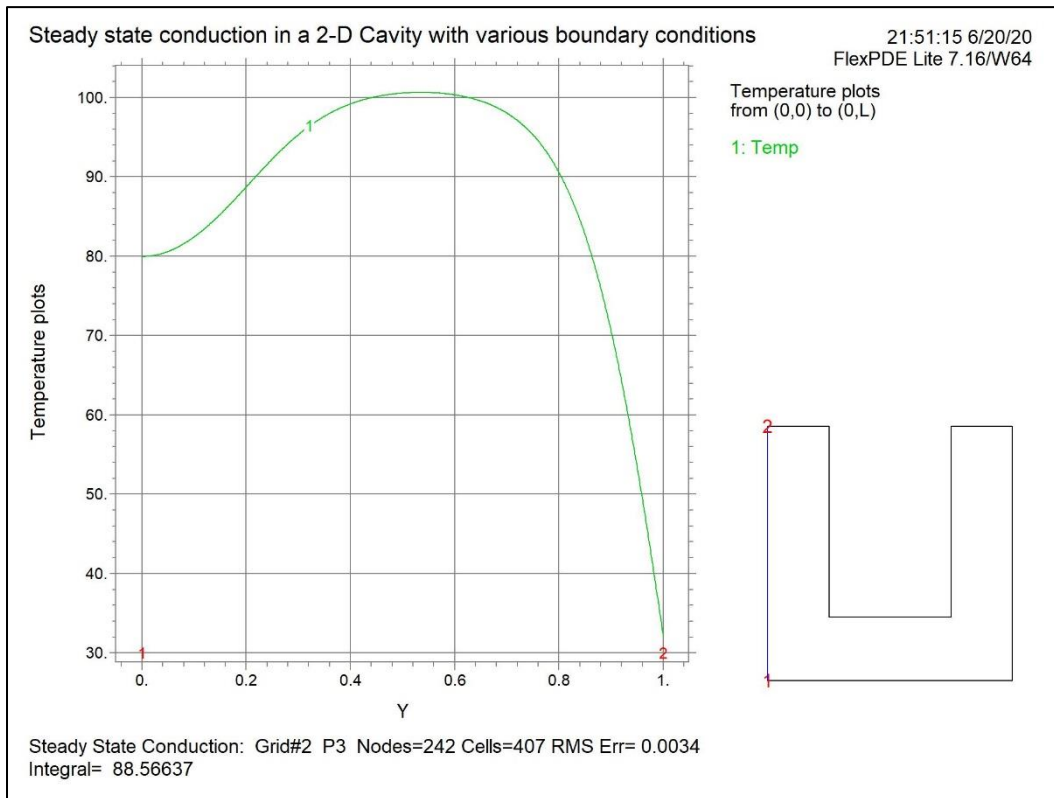


Figure 21: Elevation of the right edge from FlexPDE

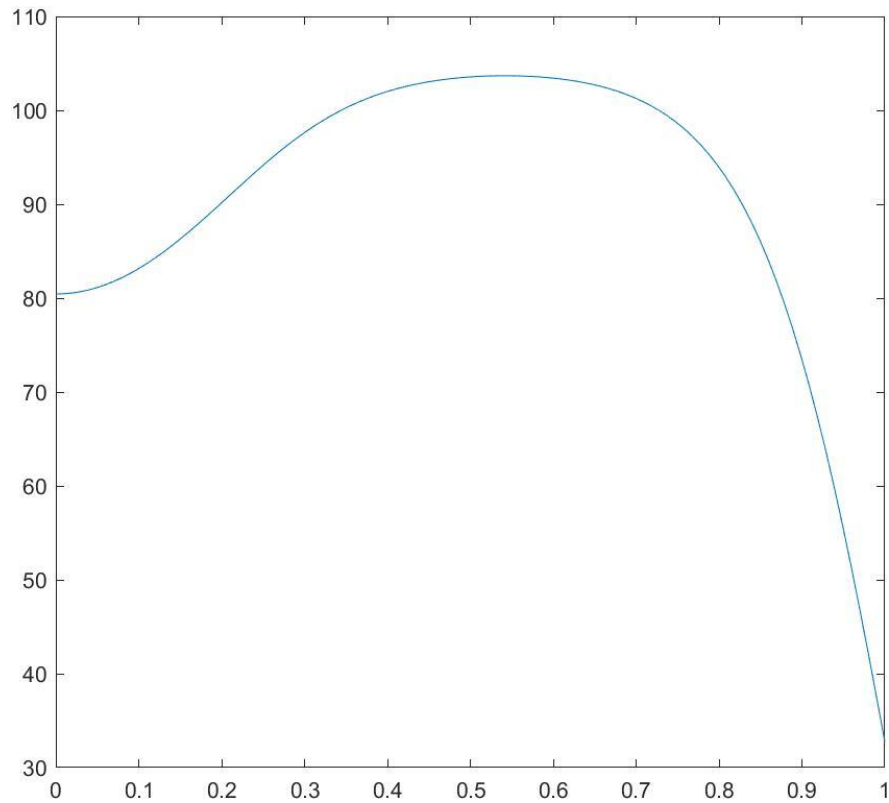


Figure 22: Elevation of the left edge from MATLAB App

Conclusion:

The FDM heat transfer model is capable of finding heat flow with different boundary conditions. From the above results of MATLAB App and results of FlexPDE, it can be concluded that all the finite difference equations derived were true and provides an approximate solution of the heat transfer for the selected domain. For the given values of the variables the temperature range is from around 30°C to 1300°C. By using the same concept heat flow can be calculated for any 2D or 3D geometry.

References:

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